One way to solve for the exact edge is by integrating the difference of the probability of a win and the probability of a loss over some subset of the unit interval. Two basic problems are that the limits of the subinterval are unknown and the probability density function is unknown.

Let $y$ = referees number (uniform on [0,1])
Let $z$ = my choice of a number from some probability density function (pdf), $f(z)$
Let $x$ = choice of opponent who selects number strategically with the knowledge that I will make an optimal choice.
Let $[u, v]$ be the unknown interval in which the pdf is positive.

We need to set the Edge using the optimal strategy to be equal to zero, because our opponent could use the same optimal strategy. Thus, the best we can hope for is to for a zero edge over an entire interval given that our opponent also chooses any point $x$ in that interval. If our opponent chooses $x$ outside that interval, we should expect a positive edge.

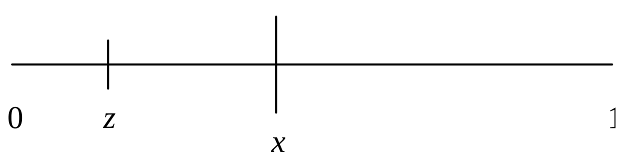
Thinking about the problem, one can realize that once you pick any strategy, a thoughtful opponent will either pick (1) $x = 0$, if, on average, your strategy selects numbers that are too high; (2) $x = v$, if on average your strategy selects numbers that are too small; or (3) the $x$ in the interval $[u,v]$ that minimizes your expected edge. If $u > 0$, an opponent's selection of zero dominates the selection of any other number between 0 and $u$ by minimizing the chance of both players going over $y$. If $v < 1$, an opponent's selection of $v$ dominates the selection of any other number between $v$ and 1, again by minimizing the chance of both players going over $y$.

Let W = Probability of win for player 1; we win when $x < z < y$ and when $z < y < x$.
Let L = Probability of loss for player 1; we lose when $x < y < z$ and when $z < x < y$.
Let T = Probability of tie for player 1; we tie when $y < x < z$ and when $y < z < x$.
And W + L + T = 1.

The edge equals W – L in this problem since ties are worth zero.

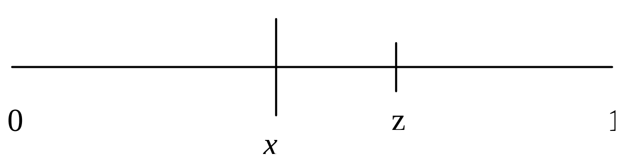
Generally we would have to solve double integrals to determine W, L, and T, integrating over all possible values of y and z, but since we know y is a uniform random variable, we can use the figures below to determine the necessary probabilities for y, depending on whether $z < x$ or $x < z$.

$z < x$:




$y$ can be greater than $x$ with probability $1 - x$, which would be a loss, and $y$ can be between $z$ and $x$ with probability $x - z$, which would be a win.

$z > x$:




$y$ can be greater than $z$ with probability $1 - z$, which would be a win, and $y$ can be between $z$ and $x$ with probability $z - x$, which would be a loss.

So, we can find an expression for the expected edge by subtracting the probability of a loss from the probability of a win over the two possibilities and equate the sum to zero. Note in the integration, we multiply the edge by the pdf to find the expected edge.

Expected edge =

$$\int_u^x f(z)[(x-z)-(1-x)]dz + \int_x^v f(z)[(1-z)-(z-x)]dz =$$

$$\int_u^x f(z)(2x-z-1)dz + \int_x^v f(z)(1-2z+x)dz = 0$$

We are sort of lucky with this problem to be able to find u and v independently of knowing f(z), simply because of the properties of pdfs.

We know the expression for the expected edge must be zero for any x in the interval; so, in particular, the expression must be true if $x = u$ or $x = v$.

First, let $x = u$ and we find that the first integral vanishes and we are left with

$$\int_u^v f(z)(1-2z+u)dz = 0$$

$$(1+u)\int_u^v f(z)dz - 2\int_u^v zf(z)dz = 0$$

Or $1 + u = 2\,E[z]$ because the first integral must equal one since this is integrating a pdf over its domain and the second integral is the definition of the expected value of $z$.

So, we have $E[z] = (1 + u) / 2$

Now, let $x = v$. The second integral vanishes and we have

$$\int_u^v f(z)(2v-z-1)dz = 0$$

$$(2v-1)\int_u^v f(z)dz - \int_u^v zf(z)dz = 0$$

Leading to $E[z] = 2v - 1$

Equating the two different right hand expressions for E[z] establishes that $v = (u + 3)/4$.

The next part is not immediately obvious, but we can take advantage of the fact that if our opponent selects x = 0, then we know the edge of the optimal strategy is not negative.

Our edge in that case would be $\int_u^v f(z)(1-2z)dz \geq 0 \Rightarrow 1-2E[z] \geq 0 \Rightarrow E[z] \leq 1/2$

We also know from above that $E[z] = (1 + u) / 2$, so if $(1 + u) / 2 = 1/2$, then we know that $u = 0$; so, since we cannot have a negative u, we have $u = 0$ and then $v$ must be 3/4.

Okay, we have our domain. Now, for the fun part; substitute the values u = 0 and v = 3/4 into our expression for the expected edge.

$$\int_0^x f(z)(2x-z-1)dz + \int_x^{3/4} f(z)(1-2z+x)dz = 0 \quad \Rightarrow$$

$$(2x-1)\int_0^x f(z)dz - \int_0^x zf(z)dz + (1+x)\int_x^{3/4} f(z)dz - 2\int_x^{3/4} zf(z)dz = 0 \Rightarrow$$

$(2x-1)F(x)-\int_0^{3/4} zf(z)dz+(1+x)(1-F(x))-\int_x^{3/4} zf(z)dz=0$, where $F(x)$ is the antiderivative of $f(x) \Rightarrow$

$(x-2)F(x)-\frac{1}{2}+1+x-\int_x^{3/4} zf(z)dz=0$ since the first integral is the expected value of z.

The second integral can be converted through integration by parts to:

$$\int_x^{3/4} zf(z)dz=\left[zF(z)\right]_{z=x}^{z=3/4}-\int_x^{3/4} F(z)dz=$$

$\frac{3}{4}-xF(x)-\left[G\left(\frac{3}{4}\right)-G(x)\right]$, where $G(x)$ is the antiderivative of $F(x)$.

Putting these together we get

$(x-2)F(x)-\frac{1}{2}+1+x-\frac{3}{4}+xF(x)+G\left(\frac{3}{4}\right)-G(x)=0$, which leads to a differential equation. We will try to solve for $G(x)$, then take the 2nd derivative to get to $f(x)$, our unknown density.

$$(2x-2)G'(x)-G(x)=\frac{1}{4}-x-G\left(\frac{3}{4}\right)$$

$G'(x)-\frac{1}{2x-2}G(x)=\frac{C_0-(x-1)}{2x-2}$, where $C_0$ is a constant term (combining the constants 1/4, $G(3/4)$ and allowing us to put the right hand side into a form with x – 1 in the numerator to facilitate integration later in the solution process.

First solve the homogeneous differential equation by direct integration:

$G'(x)-\frac{1}{2x-2}G(x)=0$ Let $w = G(x)$

$$\frac{dw}{dx}=\frac{1}{2x-2}w \quad \Rightarrow \quad \int\frac{dw}{w}=\int\frac{dx}{2x-2} \quad \Rightarrow \quad \ln w+c_1=\frac{1}{2}\ln(x-1)+c_2 \quad \Rightarrow$$

$$\ln w=\frac{1}{2}\ln(x-1)+c_2-c_1 \quad \Rightarrow \quad w=c_3\sqrt{x-1}$$

This is the homongeneous solution. Notes that all the $c$'s are constants that I can arbitrarily combine to get other constants.

The general solution to the differential equation $G'(x)-\frac{1}{2x-2}G(x)=\frac{C_0-(x-1)}{2x-2}$ is the homogeneous solution plus $\exp\left(\int\frac{dx}{2x-2}\right)\int\left(\frac{C_0-(x-1)}{2(x-1)}\right)\exp\left(-\int\frac{dx}{2x-2}\right)dx$ =

$$\exp\left(\frac{1}{2}\ln(2x-2)\right)\int\left(\frac{C_0-(x-1)}{2(x-1)}\right)\exp\left(-\frac{1}{2}\ln(2x-2)\right)dx =$$

$$\sqrt{2x-2}\int\left(\frac{C_0-(x-1)}{2(x-1)}\right)\frac{1}{\sqrt{2x-2}}dx=$$

$$\sqrt{2x-2}\left[\int\left(\frac{C_0}{2(x-1)\sqrt{2x-2}}\right)dx-\int\left(\frac{(x-1)}{2(x-1)\sqrt{2x-2}}\right)dx\right]=$$

$$\sqrt{2x-2}\left[-\frac{C_0}{\sqrt{2x-2}}-\frac{\sqrt{2x-2}}{2}\right]=-C_0-x+1$$

So, we get as our differential equation solution $G(x) = c_3\sqrt{x-1}-C_0-x+1$

First derivative is $F(x) = -\frac{c_3}{2}(x-1)^{-1/2}-1$

Second derivative is $f(x) = \frac{3c_3}{4}(x-1)^{-3/2}$ , which we want to be a density.

Integrating this over the interval of [0, 3/4], we get

$$\int_0^{3/4}\frac{3c_3}{4}(x-1)^{-3/2}dx=\left[-\frac{3c_3}{2}(x-1)^{-1/2}\right]_{x=0}^{x=3/4}=-\frac{3c_3}{2}\left(\frac{1}{\sqrt{-1/4}}-\frac{1}{\sqrt{-1}}\right)$$

Setting this result equal to 1 yields $c_3 = 2 / 3i$, which leads to the final answer, switching the argument from $x$ to $z$:

$$f(z)=\frac{1}{2(1-z)^{3/2}}$$

With that density function, the edge for the player selecting this random number is always zero if his opponent selects any number from [0,3/4] and the edge is 2x - 3/2 > 0 if his opponent selects x > 3/4.

Certainly if both players play this optimal strategy, the expected edge of each will be zero. Ironically, a player using ANY unsophisticated strategy will also have an expected edge of zero against an optimally-playing opponent IF all the choices of the unsophisticated strategy are ALWAYS in the range [0,3/4], regardless of proportion. But any such unsophisticated strategy would be vulnerable to having a negative edge against some other strategy, a weakness absent in the optimal strategy.

You can check this pdf by putting it back into our original integrals and solving for our expected edge:

$$\int_0^x\frac{2x-z-1}{2(1-z)^{3/2}}dz+\int_x^{3/4}\frac{1-2z+x}{2(1-z)^{3/2}}dz=$$
$$\left(3-2x-3\sqrt{1-x}\right)-\left(2x-3+3\sqrt{1-x}\right)=0 \checkmark$$