Instead of the poles being 50 feet high and the bottom 10 feet from the ground, let's simplify the problem by cutting the poles by 10 feet, so the bottom will touch the ground. Let's say it touches the ground at (0,0). Let's say the base of each pole is c meters away, so two other points are (c,40) and (-c,40).

A dangling rope from two fixed points will follow a curve called a catenary, otherwise known as a hyperbolic cosine. The equation for a catenary hanging down is:

y = cosh(x) = $(e^{x} + e^{-x}) / 2$.

However, we need some kind of scaling factor to have the curve go through our desired points. According to the Wikipedia entry on the catenary[1], the equation for a scaled catenary is

$$y = a \times (e^{x/a} + e^{-x/a}) / 2$$

This makes sense because if we scale both x and y by 1/a, then we would get the equation above.

For x=0, y = a, so let's subtract a, so that the curve at least goes through (0,0):

$$y = a \times (e^{x/a} + e^{-x/a}) / 2 - a$$

In other words:

$$y = a \times \cosh\left(\frac{x}{a}\right) - a$$

Next recall the equation for arc length. Let's define c, where the base of the two polls are (c,0) and (-c,0).

Arc length = $\int_{-c}^{c} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Recall the derivative of cosh(x) = sinh(x). Recall:

$$y = a \times \cosh\left(\frac{x}{a}\right) - a$$

In taking the derivative, remember to follow the chain rule and multiply by the derivative of the inside of the expression, which equals 1/a.

y' = (1/a) × a × $\sinh\left(\frac{x}{a}\right)$ = $\sinh\left(\frac{x}{a}\right)$

Getting back to the arc length:

Arc length = $\int_{-c}^{c} \sqrt{1 + \ (\sinh^2 \left(\frac{x}{a}\right)} \ dx$

Next recall:

$cosh^2$(x) - $sinh^2$(x) = 1

Or:

$sinh^2$(x) = $cosh^2$(x) - 1

So

Arc length = $\int_{-c}^{c} \sqrt{1 + \ (\cosh^2 \left(\frac{x}{a}\right) - \ 1} \ dx \ = \ \int_{-c}^{c} \sqrt{\cosh^2 \left(\frac{x}{a}\right)} \ dx$

= $\int_{-c}^{c} \cosh(\frac{x}{a}) \, dx$

Next recall the derivative of sinh(x) = cosh(x).

= a × $\sinh\left(\frac{x}{a}\right)$ from -c to c

We're given the arc length is 100 from x = -c to c, so it must be 50 from x = 0 to c:

50 = a × sinh(x/a) from 0 to c

Next recall sinh(x) = (e^x + e^{-x}) / 2

So, we have:

50 = a × ($e^{x/a}$ + $e^{-x/a}$) / 2 from 0 to c

50 = a ×($e^{c/a}$ + e^{-c} - $e^{0/a}$ + $e^{-0/a}$)

50 = a × ($e^{c/a}$ + $e^{-c/a}$ - 1 + 1)

50 = a × ($e^{c/a}$ + $e^{-c/a}$)

50 = a × sinh($\frac{c}{a}$)

Let's rearrange:

sinh($\frac{c}{a}$) = $\frac{50}{a}$

Next, let's go back to the beginning and recall the equation of our curve is

y = a × cosh$\left(\frac{x}{a}\right)$- a

The point at the top of the right pole is at (c,40). Let's plug that into our curve:

40 = a × cosh($\frac{c}{a}$) - a

Let's rearrange a bit:

cosh($\frac{c}{a}$) = $\frac{40+a}{a}$

Next, let's go back to this equation:

$cosh^2$(x) - $sinh^2$(x) = 1

In our case, x = $\frac{c}{a}$, so:

$cosh^2$($\frac{c}{a}$) - $sinh^2$($\frac{c}{a}$) = 1

Let's put our equations from above into that:

$(\frac{40+a}{a})$^2 - $(\frac{50}{a})$^2 = 1

(1600 + 80a + a^2 - 2500) / a^2 = 1

a^2 + 80a - 900 = a^2

80a = 900

a = 900/80 = 45/4

Let's go back to our equation for half the length of the rope:

50 = a × ($e^{c/a}$ + $e^{-c/a}$)

We finally know a, so we can rewrite that as:

50 = (45/4) × ($e^{4c/45}$ + $e^{-4c/45}$)

40/9 = ($e^{4c/45}$ + $e^{-4c/45}$)

One equation and one unknown, so we just need to solve for c. At this point, you could use equation solving software (I prefer Goal Seek in Excel) to solve for c, which you would be to be equal to 24.71877. The distance between the poles is twice that, for an answer of 49.43753 meters.

However, that seems so unsatisfying. Let's to find an exact expression for c. Let's try to make the right side an expression of sinh(x).

Recall sinh(x) = (e^{x} + e^{-x}) / 2

We left off with:

40/9 = ($e^{4c/45}$ + $e^{-4c/45}$)

20/9 = ($e^{4c/45}$ + $e^{-4c/45}$) / 2

40/9 = sinh(4c/45)

Next, take the inverse of both sides.

$sinh^{-1}$(40/9) = c × (4/45)

c = (45/4) $sinh^{-1}$ (40/9)

Since we're looking for the distance between the two poles, the answer is 2c:

Distance between two poles = (45/2) × $sinh^{-1}$ (40/9)

Next, recall:

$$sinh^{-1}(x) = \ln(\, x + \sqrt{1 + \, x^2})$$

Distance between two poles is (45/2) × $\ln(\, 40/9 + \sqrt{1 + \, (\frac{40}{9})^2})$

= (45/2) × ln($\frac{40}{9} + \sqrt{\frac{81}{81} + \frac{1600}{81}}$)

= (45/2) × ln($\frac{40}{9} + \sqrt{\frac{1681}{81}}$)

= (45/2) × ln($\frac{40}{9} + \frac{41}{9}$)

= (45/2) × ln(81/9)

=(45/2) × ln(9)

=(45/2) × 2 × ln(3)

=45 ×ln(3)

Footnotes:

1 Wikipedia entry on the catenary: en.wikipedia.org/wiki/Catenary

My thanks to Presh Talwalker for this problem. He solves a similar one in his video titled *Can You Solve Amazon's Hanging Cable Interview Question?***,** which can be found at https://www.youtube.com/watch?v=l_ffdarcJiQ