The question is which is greater, \(\pi ^e\) or \(e^\pi\)?

Let's start by making both exponents equal to \(e^\pi\), so we can get rid of them. To do that note that

\[
\pi ^e = (\pi^{1/\pi})^{\pi e} \quad \text{and} \quad e^n = (e^{1/e})^{ne}
\]

So, we can rewrite the question at hand to, which is greater, \((\pi^{1/\pi})^{\pi e}\) or \((e^{1/e})^{ne}\)?

Take the logarithm to the base \(\pi^e\) of both sides and we now ask which is greater, \(\pi^{(1/\pi)}\) or \(e^{(1/e)}\)?

Next, consider the question, at what value is \(x^{1/x}\) a maximum? Perhaps, if we knew the answer to that, we could see which value is closer, \(\pi^{1/\pi}\) or \(e^{1/e}\).

Let \(y = x^{1/x}\).

Let's take the log of both sides: \(\ln(y) = (1/x)*\ln x\).

Next, take the derivative of both sides with respect to \(x\): \(y' * (1/y) = (1/x)*(1/x) - (1/x)^2 * \ln(x)\)

Keep going: \(y' = (1/x)^2*(1-\ln(x)) * x^{1/x}\)

Let's set that equal to zero to find where the curve is a maximum or minimum.

\[(1/x)^2*(1-\ln(x)) * x^{1/x} = 0\]

\((1/x)^2\) and \(x^{1/x}\) are never going to be zero, so the \((1-\ln(x))\) must be term equal to 0:
\begin{align*}
1-\ln(x) &= 0 \\
\ln(x) &= 1 \\
x &= e
\end{align*}

So, \(e^{1/e}\) is either a maximum or minimum of the function \(x^{1/x}\)? Let's put in values of \(x\) in the equation for \(y'\) to see the derivative curve looks to the left and right of \(x=e\).

Again, \(y' = \frac{1}{x^2}(1-\ln(x)) \cdot x^{1/x}\)

What if \(x=1\)? Then \(y' = (1/1)^2(1-\ln(1))\cdot1^{1/1} = 1\cdot1\cdot1 = 1\).

What if \(x=e^2\)? The first and last terms will be hard to evaluate exactly, but all we need to know is if the whole expression is positive or negative. \((1/x)^2\) and \(x^{1/x}\) will both obviously be positive at \(x=e^2\). The middle term, \(1-\ln(x) = 1-\ln(e^2) = 1-2\cdot\ln(e) = 1-2 = -1\). Thus the whole expression of \(y'\) will be negative at a value of \(e^2\).

So, if the derivative curve is positive before \(x=e\) and negative after, then \(x=e\) must be a relative maximum.

Thus, the function \(y = x^{1/x}\) reaches a maximum value at \(x=e\).

So \(e^{1/e} > = \pi^{1/\pi}\)

Going back to the beginning, if \(e^{1/e} > \pi^{1/\pi}\), then \(e^{\pi} > \pi^e\)