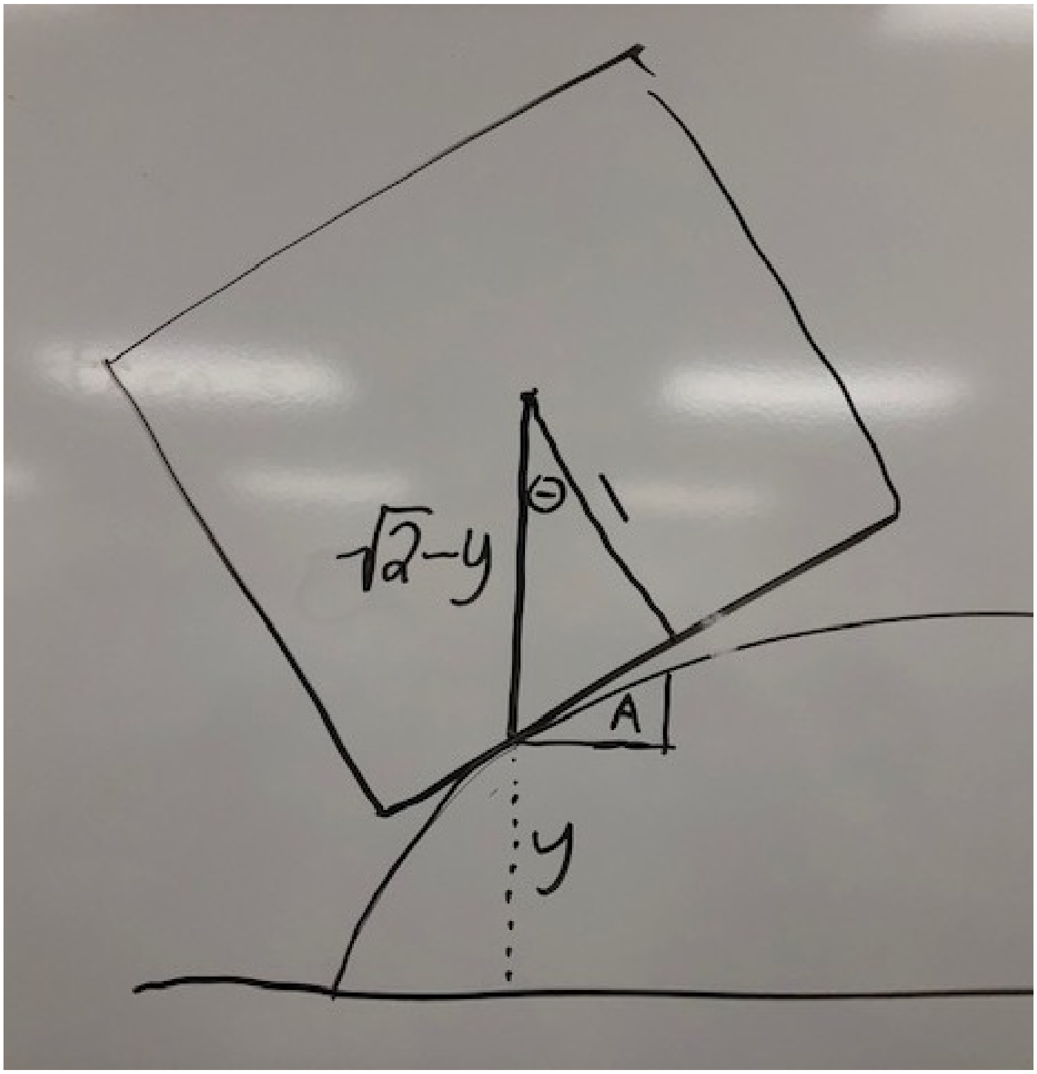
National Museum of Mathematics -- New York, New York

Question: A square wheel travels over a rounded track so that the center is always at the same height. The sides of the wheels have length 2. If the square makes a full rotation, how much horizontal distance does it cover?

Hints:

$(cosh^{-1}(x))' = \frac{1}{\sqrt{x^2 - 1}}$

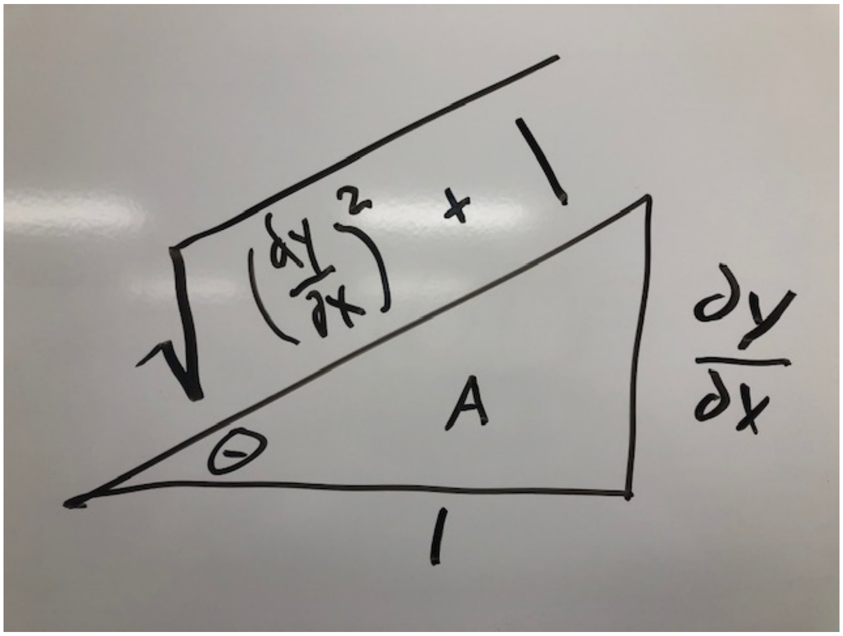
$\cosh^{-1}(x) = \ln(x \pm \sqrt{x^2 - 1})$



Let (x,y) be the coordinates of the point where the wheel touches the track. The diagram above shows an example of a position of the wheel on the track. The least y can be is 0 when one corner is directly above the opposite corner. By the Pythagorean theorem, the distance from the center from the x axis in this position is $\sqrt{2}$. Thus, the center of the wheel will always be at a height equal to $\sqrt{2}$.

It stands to reason that the center of the wheel will always be directly above the point of contact with the track. By the Pythagorean theorem, the vertical distance from y to the center of the wheel at any given time will be $\sqrt{2}$ - y.

Next, let's focus on the small triangle A in the first image, which represents an incremental change as the wheel moves to the right.



For an incremental horizontal movement of 1 (not to be confused with the distance of the center of the square to the edge), the change in the vertical movement will be dy/dx. By the Pythagorean theorem, the change in the total distance of the point of contact to the curve will be:

$$\sqrt{(\frac{dy}{dx})^2 + 1}$$

Simple geometry will show that θ from both triangles are the same.

Next, let's equalize the secant of θ in both triangles.

$\sqrt{2}$ - y = $\sqrt{(\frac{dy}{dx})^2 + 1}$

Let's solve for dy/dx:

$(\sqrt{2} - y)^2 = (\frac{dy}{dx})^2 + 1$

$(\frac{dy}{dx})^2 = (\sqrt{2} - y)^2 - 1$

$\frac{dy}{dx} = \sqrt{(\sqrt{2} - y)^2 - 1}$

Let's break up that dy/dx :

$\frac{1}{\sqrt{(\sqrt{2} - y)^2 - 1}}$ dy = 1 dx

Let's let u = $\sqrt{2} - y$
du = - dy
dy = -du

$\frac{-1}{\sqrt{u^2 - 1}}$ du = 1 dx

Recall the first hint:

$(cosh^{-1}(x))' = \frac{1}{\sqrt{x^2 - 1}}$

Integrating both sides we get:

$\cosh^{-1}(u) + c = x$

$\cosh^{-1}(\sqrt{2} - y) + c = x$

Let's solve for the constant of integration c. Consider the situation where the square is on the top of the curve. Let's set that as the starting point, where x=0. As stated before, the distance from the x-axis to the center of the circle is sqrt(2), so at x=0, y = sqrt(2) - 1. Putting that point into our equation:

$\cosh^{-1}(\sqrt{2} - (\sqrt{2} - 1)) + c = 0$

$\cosh^{-1}(1) + c = 0$

$\cosh^{-1}(1) = -c$

Take the cosh() of each side:

$1 = \cosh(-c)$

$1 = (e^{-c} + e^{c})/2$

$2 = e^{-c} + e^{c}$

This can be true at c=0 only, thus c = 0.
Now our equation is:

$\cosh^{-1}(\sqrt{2} - y) = x$

Recall that the square started flat, at x=0. In 1/8 of a revolution, y will be 0, and x will be the horizontal distance covered:

$x = \cosh^{-1}(\sqrt{2})$

Recall our second hint:

$\cosh^{-1}(x) = \ln(x \pm \sqrt{x^2 - 1})$

$x = \ln(\sqrt{2} \pm \sqrt{2 - 1})$

$x = \ln(\sqrt{2} + 1) \approx 0.8814$

The question asked what is the horizontal distance covered with a full revolution, which will be eight times that of 1/8 of a revolution, or:

$8 \times \ln(\sqrt{2} + 1) \approx 7.0510$

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