Question: In the following diagram, what is the area of the green region?

Answer:

First, let's change the dimensions to $2 \times 1$, to simplify the math, reducing the scale of the area by a factor of $10 \times 5 / 2 \times 1 = 25$. We will multiply by 25 at the end to scale back up. Second, let's flip over the image, for reasons we'll see soon. That gives us:
Let's use geometry to solve the problem. Let's place the center of the bottom of the rectangle at coordinate (0,0).

The equation for the circle is \( x^2 + y^2 = 1 \). This illustrates why I wanted to scale down and flip, to get such a simple equation for the circle.

The equation of the blue line is \( y = x/2 + 1/2 \). Let's rewrite that as \( x = 2y - 1 \).

Let's solve for \( x \) and \( y \) to find where the blue line intersects the semicircle.

\[
\begin{align*}
x^2 + y^2 &= 1 \\
(2y - 1)^2 + y^2 &= 1 \\
5y^2 - 4y &= 0 \\
5y - 4 &= 0 \\
y &= 0.8
\end{align*}
\]

Putting that in \( x=2y-1 \) gives us

\[x = 2\times0.8 -1 = 0.6.\]

Next, let's label some of the regions in play.
Let's start by finding the slice of the semicircle identified as \( C + D \). We already know the coordinate where the blue line crossed the semicircle by the green region is \((0.6, 0.8)\). So, the side of triangle \( D \) are 0.6, 0.8, and 1. The area of \( D \) is easily found as \( \frac{1}{2} \times (0.6 \times 0.8) = 0.24 \).

To find \( C \), let's find the area of the slice of the semicircle \( C+D \) and subtract \( D \) from it.

The angle of \( D \) at \((0,0)\) can be expressed as \( \tan^{-1}(3/4) \), \( \cos^{-1}(4/5) \), or \( \sin^{-1}(3/5) \). I'll arbitrarily decide to go with \( \cos^{-1}(4/5) \approx 0.6435 \) (in radians).

The area of the whole circle is \( \pi \), divided up by \( 2\pi \) radians, so the area of the circle \( (C+D) \) formed by an angle of \( \cos^{-1}(4/5) \) radians is \( \frac{\cos^{-1}(4/5)}{2} \approx 0.3218 \).

We subtract \( D \) from that slice to get the area of \( C = \frac{\cos^{-1}(4/5)}{2} - 0.24 \approx 0.0818 \).

The area of rectangle \( A+C \) is \( 0.2 \times 0.6 = 0.12 \). We know \( C \), so we can find \( A \) as \( 0.12 - [\cos^{-1}(4/5)/2 - 0.24] = 0.36 - \cos^{-1}(4/5) \approx 0.0382 \).

The two legs of triangle \( B \) are 0.2 and 0.4, thus the area of \( B \) is \( \frac{0.2 \times 0.4}{2} = 0.04 \).
Thus, the green region is $A + B = 0.36 - \cos^{-1}(4/5) + 0.04 = 0.4 - \cos^{-1}(4/5)/2 = \approx 0.0782494$

Remember we scaled the problem down by a factor of 25 at the beginning, so let’s scale that area up by a factor of 25, to account for the 5x10 region to begin with, to get an answer of $10 - 12.5 \times \cos^{-1}(4/5) = \approx 1.95624$

My thanks to Presh Talwalker for this problem, who in turn gives credit to Xavier in Shanghai. Presh’s YouTube channel is Mind Your Decisions. He goes over a solution to this problem at https://www.youtube.com/watch?v=2Seb863FnfU

Michael Shackleford
MathProblems.info
April 20, 2019