Question: A plane flies at 10 km./minute in a straight direction and level altitude. When it is directly 5 km. overhead you shoot a heat-seeking missile at it, straight up. The missile always points directly at the plane. The missile travels at a constant speed of 11 km./minute. How long does it take to hit the plane?

Hint: The integral of $\sqrt{1 + u^2}$ du = ln(u + $\sqrt{1 + u^2}$) + c, where c is the constant of integration.

Solution

Assume the shooter is at point (0,0) and the plane is at (0,5) when the missile is fired, traveling easy on the coordinate plane.

Let
x = x-coordinate of missile.
y = y-coordinate of missile.
t = number of minutes since missile was fired.
s = distance traveled by missle.

At all times the slope of the missile = $\frac{dy}{dx} = \frac{5-y}{10t-x}$

Invert that:

$$\frac{dx}{dy} = \frac{10t-x}{5-y}$$

Let's put dy and (5-y) on the same side.

(5-y) $\frac{dx}{dy}$ = 10t-x

Take the derivative of both sides with respect to y. Use the product rule on the left side.

- $\frac{dx}{dy}$ + (5-y) $\frac{d^2x}{dy^2}$ = 10 $\frac{dt}{dy}$ - $\frac{dx}{dy}$

[1] (5-y) $\frac{dx^2}{dy^2}$ = 10 $\frac{dt}{dy}$

Let $\frac{dt}{dy}$ = $\frac{dt}{ds} \circ \frac{ds}{dy}$

$\frac{ds}{dt}$ = 11, so $\frac{dt}{ds}$ = $\frac{1}{11}$

The equation for the arc-length of s is:

s = $\int \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

$\frac{ds}{dy}$ = $\sqrt{1 + (\frac{dx}{dy})^2}$

Let's go back to equation [1]:

[1] (5-y) $\frac{dx^2}{dy^2}$ = 10 $\frac{dt}{dy}$

(5-y) $\frac{dx^2}{dy^2}$ = $\frac{10}{11}$ × $\frac{ds}{dy}$

(5-y) $\frac{dx^2}{dy^2}$ = $\frac{10}{11}$ × $\sqrt{1 + (\frac{dx}{dy})^2}$

Let u = $\frac{dx}{dy}$

(5-y) $\frac{du}{dy}$= $\frac{10}{11}$ × $\sqrt{1 + \mathrm{u}^2}$

Let's put the expressions with a u on one side.

$\sqrt{1 + \mathrm{u}^2}$ *du* = $\frac{10}{11}$ × $(5 - y)^{-1}$ *dy*

Using the hint from the integral table:

ln(u + $\sqrt{1 + \mathrm{u}^2}$) = - $\frac{10}{11}$ × ln(5-y) + c

Take exp() of both sides:

u + $\sqrt{1 + \mathrm{u}^2}$ = $(5 - y)^{-\frac{10}{11}}$ × c

Let's solve for c. We know that at t=0 that $\frac{dx}{dy}$ = 0.

0 + $\sqrt{1 + 0^2}$ = $(5 - 0)^{-\frac{10}{11}}$ × c

c = $5^{\frac{10}{11}}$

u + $\sqrt{1 + u^2}$ = $(5 - y)^{-\frac{10}{11}} \times 5^{\frac{10}{11}}$

u + $\sqrt{1 + u^2}$ = $(\frac{5-y}{y})^{-\frac{10}{11}}$

$\sqrt{1 + u^2}$ = $(\frac{5-y}{y})^{-\frac{10}{11}}$ - u

Square both sides.

1 + u^2 = $(\frac{5-y}{y})^{-\frac{20}{11}}$ - 2u $(\frac{5-y}{y})^{-\frac{10}{11}}$ + u^2

1 = $(\frac{5-y}{y})^{-\frac{20}{11}}$ - 2u $(\frac{5-y}{y})^{-\frac{10}{11}}$

2u $(\frac{5-y}{y})^{-\frac{10}{11}}$ = $(\frac{5-y}{y})^{-\frac{20}{11}}$ - 1

Remember that u = $\frac{dx}{dy}$

$\frac{dx}{dy}$ = 0.5 × [$(\frac{5-y}{y})^{-\frac{10}{11}} - (\frac{5-y}{y})^{\frac{10}{11}}$]

Take the integral of both sides.

x = 0.5 × [$-55 * (\frac{5-y}{y})^{\frac{1}{11}} + \frac{55}{21} (\frac{5-y}{y})^{\frac{21}{11}}$] + c

Let's solve for c, using the fact that when x=0, y=0.

$0 = 0.5 \times [\ -55 + \frac{55}{21}\ ] + c$

$c = \frac{550}{21}$

So, the equation of the path of the missile is:

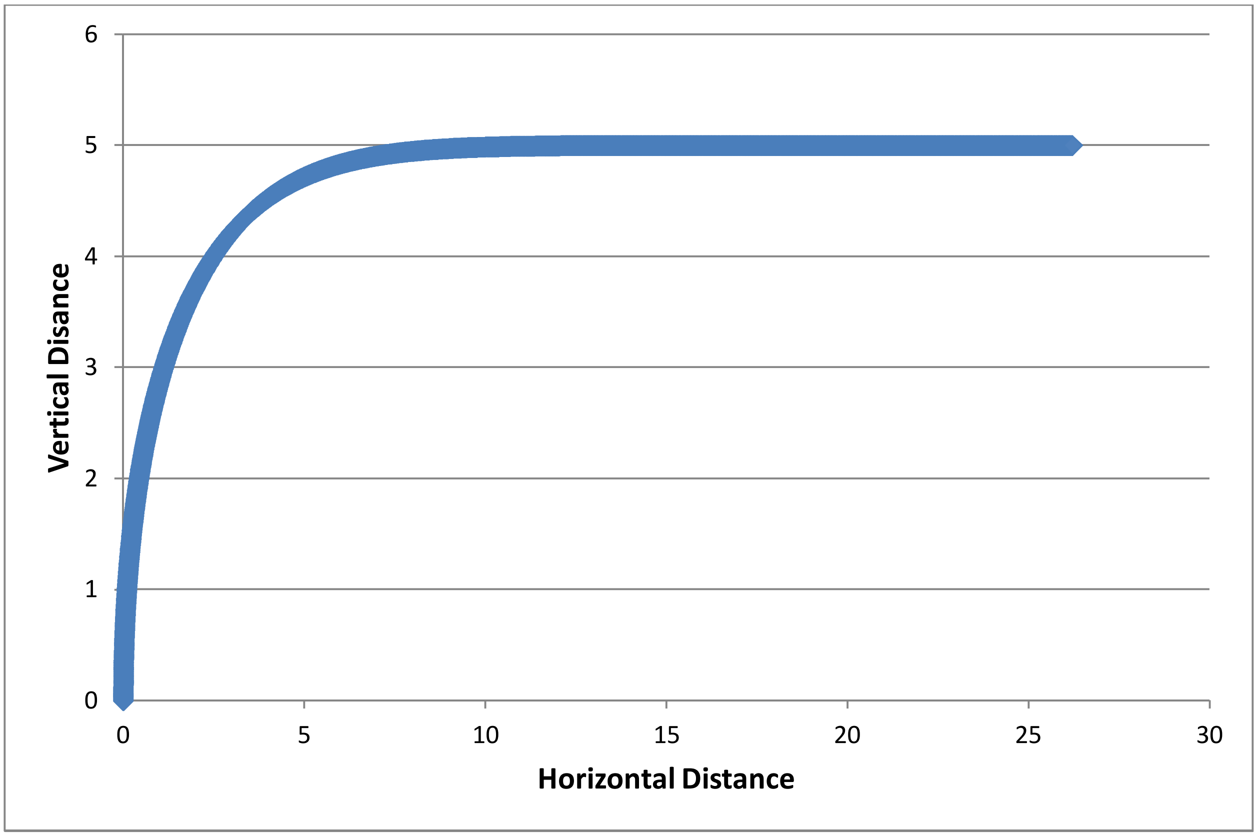
$x = 0.5 \times [-55 * (\frac{5-y}{y})^{\frac{1}{11}} + \frac{55}{21}\ (\frac{5-y}{y})^{\frac{21}{11}}\ ] + \frac{550}{21}$

The missile hits the plane when y=5. So solve for x when y=5:

$x = 0.5 \times [\ 0 + 0\ ] + \frac{550}{21}$

So, the missile strikes the plane when the plane has traveled 550/21 kilometers. The plane travels at 10 kilometers per minute, so it would have taken 55/21 = 2.6195 minutes = 157.1429 seconds to get that far.

This graph shows the path of the missile. The end at (26.195,5) is where it strikes the plane.



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