Points on a Circle

Problem

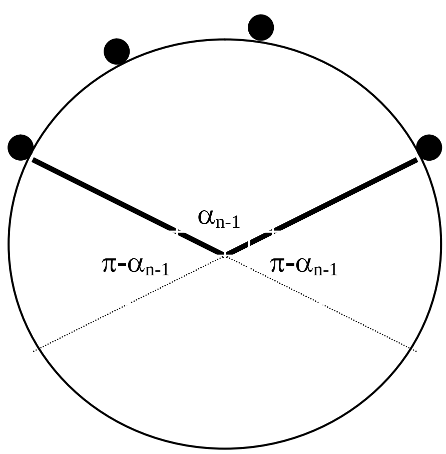
We randomly distribute n points on the circumference of a circle. What is the probability that they will all fall in a common semi-circle?

Solution

We can solve the problem by induction. Given any n-1 points on a circle, let α_{n-1} be the smallest radial angle that subtends them. Let X_n be the event that α_n and its predecessors are all less than π:

$$X_n : \{\alpha_n \le \pi,\ \alpha_{n-1} \le \pi,\ \ldots\alpha_3 \le \pi,\}$$

We are looking for the probability of X_n. We will start with the conditional probability of X_n given X_{n-1}. It is clear from the diagram that X_n will occur if α_n falls in any of the top three segments, but not if it falls in the lower segment.



$$\mathrm{Prob}(X_n \mid X_{n-1}) = \frac{2\pi - \alpha_{n-1}}{2\pi} = 1 - \frac{\alpha_{n-1}}{2\pi} \tag{1}$$

Consider next the integral of α_n over the range of possible α_n given X_{n-1}, i.e. over the top three segments of the circle. The expected value of α_n if the n^{th} point falls in the top segment is α_{n-1}. The expected value of α_n if the n^{th} point falls in either of the two side segments is the average of its values at the endpoints of those segments or $(\pi+\alpha_{n-1})/2$. Multiply each of these expected values times the probability of their occurrence:

$$\left[\frac{\alpha_{n-1}}{2\pi}\right](\alpha_{n-1}) + \left[\frac{2(\pi - \alpha_{n-1})}{2\pi}\right]\left(\frac{\pi + \alpha_{n-1}}{2}\right) = \frac{\alpha_{n-1}^2 + \pi^2 - \alpha_{n-1}^2}{2\pi} = \frac{\pi}{2} \tag{2}$$

It is easy to find the probability of X_3 from (1) since the expected value of α_2 over its full range $[0\ldots\pi]$ is $\pi/2$:

$$\mathrm{Prob}(X_3) \;=\; 1 - \frac{1}{2\pi}\frac{\pi}{2} \;=\; \frac{3}{4}$$

The probability of X_4 is only slightly harder. By (1) above:

$$\mathrm{Prob}(X_4 \mid X_3) \;=\; 1 - \frac{\alpha_3}{2\pi}$$

Integrating over $\alpha_3 \le \pi$ and invoking (2) above

$$\mathrm{Prob}(X_4) \;=\; \frac{3}{4} - \frac{1}{2\pi}\frac{\pi}{2} \;=\; \frac{1}{2} \;=\; \frac{4}{8}$$

These first two n suggest that the answer might have the form $\boxed{\mathrm{Prob}(X_n) = \frac{n}{2^{n-1}}}$ (3)

More generally, α_2 is uniformly distributed over $[0\ldots\pi]$, but the other α have conditional probabilities as follows:

$$p(\alpha_k \mid \alpha_{k-1}) = \begin{cases} \dfrac{\alpha_{k-1}}{2\pi} \text{ point probability on } \alpha_{k-1} \\ \dfrac{2}{2\pi} \text{ uniform on } [\alpha_{k-1} \ldots \pi] \end{cases}$$

$$\mathrm{Prob}(X_n) \;=\; \int_0^\pi \int_{\alpha_2}^\pi \cdots \int_{\alpha_{n-2}}^\pi \int_{\alpha_{n-1}}^\pi p(\alpha_2)\,p(\alpha_3 \mid \alpha_2)\ldots p(\alpha_{n-1} \mid \alpha_{n-2})\,p(\alpha_n \mid \alpha_{n-1})\,d\alpha_n d\alpha_{n-1}\ldots d\alpha_3 d\alpha_2$$

$$= \int_0^\pi \int_{\alpha_3}^\pi \cdots \int_{n-2}^\pi \int_{n-1}^\pi p(\alpha_2)\,p(\alpha_3 \mid \alpha_2)\ldots p(\alpha_{n-2} \mid \alpha_{n-3})\,p(\alpha_{n-1} \mid \alpha_{n-2})\left[1 - \frac{\alpha_{n-1}}{2\pi}\right] d\alpha_{n-1} d\alpha_{n-2}\ldots d\alpha_3 d\alpha_2$$

$$= \mathrm{Prob}(X_{n-1}) - \frac{1}{2\pi}\frac{\pi}{2}\int_0^\pi \int_{\alpha_3}^\pi \cdots \int_{\alpha_{n-3}}^\pi p(\alpha_2)\,p(\alpha_3 \mid \alpha_2)\ldots p(\alpha_{n-2} \mid \alpha_{n-3})\,d\alpha_{n-2}\ldots d\alpha_3 d\alpha_2$$

$$= \mathrm{Prob}(X_{n-1}) - \frac{1}{4}\mathrm{Prob}(X_{n-2})$$

This recurrence relationship allows us to prove (3) by induction. We know that (3) holds for n=3 and n=4. Suppose it holds generally for numbers up to n-1. Then the recurrence relation above becomes:

$$\mathrm{Prob}(X_n) \;=\; \frac{n-1}{2^{n-2}} - \frac{1}{4}\frac{n-2}{2^{n-3}} \;=\; \frac{2n-2-n+2}{2^{n-1}} \;=\; \frac{n}{2^{n-1}}$$

which establishes (3) as the general solution.