Question: What is the mean distance between two random points in a unit square?

Hint: $\int sec^3(x)\ dx = \frac{\sec(x)\tan(x)}{2} + \frac{\ln(\sec(x)+\tan(x))}{2} + C$

Answer:

One could express the answer as a quadruple integral over the x and y values of point points. However, trust me, it would get ugly fast.

I suggest instead expressing the answer as a double integral over the difference between the two x values and two y values.

Given two random points between 0 and 1, simple geometry will show us that the probability the distance between them is less than x equals $2x - x^2$.

The density function for the distance between the two point is the derivative of that, or 2*(1-x).

Let:
$\Delta x$ = difference in x values.
$\Delta y$ = difference in y values.

Of course, the distance between the two points can be expressed as

$$\sqrt{\Delta x^2 + \Delta y^2}$$

So an express of the answer is

$\iint_0^1 \sqrt{\Delta x^2 + \Delta y^2} * 2(1-x) * 2 * (1-y)\ dx\ dy$ =

$4 \iint_0^1 \sqrt{\Delta x^2 + \Delta y^2}\,(1-x)(1-y)\,dx\,dy$ =

Next make the following substitution:

x = r cos θ
y = r sin θ

We now have:

$4 \iint_0^{sec\theta} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} * (1 - rcos\theta)(1 - rsin\theta)\ J\ dr\ d\theta$ , where θ ranges from 0 to π/2.

Next we divide the area of integration over θ by 2 and multiply the result by 2:

$8 \iint_0^{sec\theta} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} * (1 - rcos\theta)(1 - rsin\theta)\ J\ dr\ d\theta$ , where θ ranges from 0 to π/4.

Next, let's find the Jacobian, J:

J = $\begin{matrix} dx/dr & dx/d\theta \\ dy/dr & dy/d\theta \end{matrix}$

= $\begin{matrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{matrix}$

= $r*\cos^2\theta + r*\sin^2\theta = r$

Let's solve the original equation!

$8 \iint_0^{sec\theta} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} * (1 - rcos\theta))(1 - rsin\theta)\ J\ dr\ d\theta$ =

$8 \iint_0^{sec\theta} r^2 * (1 - rcos\theta)(1 - rsin\theta)\ dr\ d\theta$ =

$8 \iint_0^{sec\theta} r^2 * (1 - rsin\theta - r\cos\theta + r^2 sin\theta\cos\theta)\, dr\, d\theta$ =

$8 \iint_0^{sec\theta} r^2 - r^3 sin\theta - r^3\cos\theta + r^4 sin\theta\cos\theta\, dr\, d\theta$ =

$8 \int_0^{\pi/4} r^3/3 - (r^4/4) *$ (sinθ + cosθ) + $(r^5/5) *$ ( sin θ * cos θ) dθ from 0 to secθ =

$8 \int_0^{\pi/4} \sec^3\theta/3 - \sec^4\theta/4$ * (sinθ + cosθ) + $(\sec^5\theta/5) *$ (sin θ * cos θ) dθ =

$8 \int_0^{\pi/4} \frac{\sec^3\theta}{3} - \frac{\tan\theta\sec^3\theta}{4} - \frac{\sec^3\theta}{4} + \frac{\tan\theta\sec^3\theta}{5}\, d\theta$ =

(It goes without saying that d/dθ $\sec^3\theta$ = 3tan θ $\sec^3\theta$)

$8 \int_0^{\pi/4} \frac{\sec^3\theta}{12} - \frac{\tan\theta\sec^3\theta}{20}\, d\theta$ =

8*(secθtanθ/24 + ln(secθ+tanθ)/24 - $\sec\theta^3$/60) from 0 to π/4 =

secθtanθ/3 + ln(secθ+tanθ)/3 - 2/15$\sec^3\theta$ from 0 to π/4 =

$\sqrt{2}$/3 + ln($\sqrt{2}$ + 1)/3 - $4\frac{\sqrt{2}}{15} + \frac{2}{15} =$

$(\sqrt{2} + \ln(1 + \sqrt{2}) + 2)/15$ =~ 0.521405433164721

My thanks to Presh Talwalker for this solution.

Link to his solution: https://www.youtube.com/watch?v=i4VqXRRXi68