Coin Toss Problem

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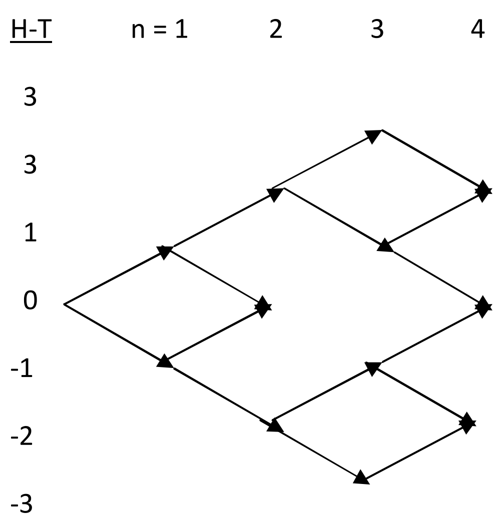
Problem:

Given a coin with probability p of landing on heads after a flip, what is the probability that the number of heads will ever equal the number of tails assuming an infinite number of flips?

Solution:

We are interested in H-T, the number of heads minus the number of tails. After one flip, H-T will be either 1 or -1. Beyond this, pathways branch out; some will reach equality and perhaps some not. To avoid double-counting, we must terminate all paths that achieve equality, observing what fraction of the possible paths we terminate. The H-T>0 process is separated from the H-T<0 process, as follows:



Let $q = 1-p$. The transition probability matrix gives the probability of any state i moving to any other state j with one more flip. The rows represent present state and the columns represent the next state. For the H-T>0 process, the transition probability matrix P is:

To: From	**0**	**1**	**2**	**3**	**4**	**5**
0	1	0	0	0	0	0
1	q	0	p	0	0	0
2	0	q	0	p	0	0
3	0	0	q	0	p	0
4	0	0	0	q	0	p

This says that from equality (state 0) there is no probability of moving to any other state: 0 is an absorbing barrier. From state 1 there is a q probability of moving to the absorbing barrier, no probability of staying in state 1 and a p probability of moving to state 2. All rows sum to 1. The transition probabilities for the H-T<0 process are the mirror image, with p and q reversed.

When either process starts at ±1, how does it evolve? Using the rules of matrix multiplication, multiply P by itself n times, the resulting matrix P^n represents the probability of moving from any state i to any state j in n moves. As n increases without limit, P^n evolves toward a limiting matrix L. Because of the absorbing barrier at 0, L has this form:

To: From	**0**	**1**	**2**	**3**	**4**	**5**
0	1	0	0	0	0	0
1	L_1	0	0	0	0	0
2	L_2	0	0	0	0	0
3	L_3	0	0	0	0	0
4	L_4	0	0	0	0	0

The limiting matrix must have the property that L = PL, i.e. multiplying L once more by P will not change it. Referring back to the definition of P, this means the L_i must satisfy these equations:

$$L_1 = q + p\,L_2$$

$$L_2 = q\,L_1 + p\,L_3$$

$$L_3 = q\,L_2 + p\,L_4$$

....

$L_i = q\ L_{i-1} + p\ L_{i+1}$

Since these are all of the same form, there must be some regularity among the L_i. Let us guess that this regularity is a simple ratio: $L_1 = kL_0$, $L_2 = kL_1$, $L_3 = kL_2$, . . . $L_i = kL_{i-1}$. . . Then the last equation becomes:

$L_i = q\ L_i/k + p\ kL_i$

Cancelling the L_i and rearranging,

$p\ k^2 - k + q = 0$

This quadratic equation has two roots: $k = q/p$ and $k = 1$. They lead to two different limiting matrices. The entries in a valid limiting matrix are probabilities and so are bounded by 0 and 1. The first k can apply only when $q<p$, so L that becomes:

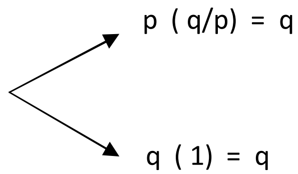
From \ To:	**0**	**1**	**2**	**3**	**4**	**5**
0	1	0	0	0	0	0
1	**q/p**	0	0	0	0	0
2	q^2/p^2	0	0	0	0	0
3	q^3/p^3	0	0	0	0	0
4	q^4/p^4	0	0	0	0	0

When $q>p$, the above is not a valid probability matrix so the second k must apply, and L becomes:

From \ To:	**0**	**1**	**4**	**3**	**4**	**5**
0	1	0	0	0	0	0
1	**1**	0	0	0	0	0
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	1	0	0	0	0	0

When $p=q$ these matrices are identical.

The first matrix says that over an infinite number of flips the probability of moving from state 1 to equality is q/p. The second matrix says that this probability is 1. We now have a complete picture of the problem. Assuming $q<p$, the two branches result in the same probability of ever reaching equality:



The final answer is therefore 2q.