Problem 237 solution

Question: Three ants go around the circumference of a circle, each at his own contant rate and in the same direction. It takes ant A three minutes to make a revolution, ant B five minutes, and ant C seven minutes. They all start out at random points. A point is picked at random on the circumference. What is the probability each ant reaches that point first?

I'll present two different methods to solve this problem. Of course, I'll start with the harder calculus solution first.

Let's first answer the question of probability that ant A arrives first. Let x be the number of minutes the ant in quesiton is away from arriving. The answer can be expressed as:

$\frac{1}{3}\int_0^3 \Pr(A\ beats\ B) * \Pr(A\ beats\ C)\quad dx$ =
$\frac{1}{3}\int_0^3 \frac{5-x}{5} \times \frac{7-x}{7}\ dx$ =
$\frac{1}{105}\int_0^3 x^2 - \ 12x + 35dx$ = $\frac{4}{7}$

Probability B arrives first =
$\frac{1}{5}\int_0^3 \Pr(B\ beats\ A) * \Pr(B\ beats\ C)\quad dx$ =
$\frac{1}{5}\int_0^3 \frac{5-x}{3} \times \frac{7-x}{7}\ dx$ =
$\frac{1}{105}\int_0^3 x^2 - \ 5x + 21dx$ = $\frac{9}{35}$

Probability C arrives first =
$\frac{1}{7}\int_0^3 \Pr(C\ beats\ A) * \Pr(C\ beats\ B)\quad dx$ =
$\frac{1}{7}\int_0^3 \frac{5-x}{5} \times \frac{3-x}{3}\ dx$ =
$\frac{1}{105}\int_0^3 x^2 - \ 8x + 15dx$ = $\frac{6}{35}$

Next, let's solve this using more basic math.

After the destination point is randomly picked, there are four possible states the ants could be in, as follows:

1. Ants B and C are both more than three minutes away from arriving.
2. Ant B is less than three minutes away and C is more.
3. Ant B is more than three minutes away and C is less.
4. Ants B and C are both less than three minutes away.

Next, let's calculate the probability of each state:

1. (2/5) * (4/7) = 8/35
2. (3/5) * (4/7) = 12/35
3. (2/5) * (3/7) = 6/35
4. (3/5) * (3/7) = 9/35

For any given state, each ant that is within 3 minutes of arriving first will have an equal chance of arriving first. Note that being in the same state doesn't mean being in the same sector of the circle. It just means that whatever their starting point is, they will arrive at the destination within three minutes. Thus, all ants in a state will have an equal chance.

The following table shows the probability of each ant winning, given each state:

State	Pr (A wins)	Pr(B wins)	Pr (C wins)
1	1	0	0
2	1/2	1/2	0
3	1/2	0	1/2
4	1/3	1/3	1/3

The probability of each ant winning is the dot product of the probability of being in any given state and the probability of winning that state. Thus,

Pr (A wins) = (8/35) * 1 + (12/35) * (1/2) + (6/35)*(1/2) + (9/35)*(1/3) = 4/7
Pr (B wins) = (8/35) * 0 + (12/35) * (1/2) + (6/35)*0 + (9/35)*(1/3) = 9/35
Pr (C wins) = (8/35) * 0 + (12/35) * 0 + (6/35)*(1/2) + (9/35)*(1/3) = 6/35

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