Problem 232 Solution

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The question is which is greater, π^e or e^π?

Let's start by making both exponents equal to e*π, so we can get rid of them. To do that note that

$\pi^e = (\pi^{1/\pi})^{\pi e}$ and $e^\pi = (e^{1/e})^{\pi e})$

So, we can rewrite the question at hand to, which is greater, $(\pi^{1/\pi})^{\pi e}$ or $(e^{1/e})^{\pi e})$?

Take the logarithm to the base π*e of both sides and we now ask which is greater, $\pi^{(1/\pi)}$ or $e^{(1/e)}$?

Next, consider the question, at what value is $x^{1/x}$ a maximum? Perhaps, if we knew the answer to that, we could see which value is closer, $\pi^{1/\pi}$ or $e^{1/e}$.

Let $y = x^{1/x}$.

Let's take the log of both sides: $\ln(y) = (1/x)*\ln x$.

Next, take the derivative of both sides with respect to x: $y' * (1/y) = (1/x)*(1/x) - (1/x)^2 * \ln(x)$

Keep going: $y' = (1/x)^2*(1-\ln(x)) * x^{1/x}$

Let's set that equal to zero to find where the curve is a maximum or minimum.

$(1/x)^2*(1-\ln(x)) * x^{1/x} = 0$

$(1/x)^2$ and $x^{1/x}$ are never going to be zero, so the $(1-\ln(x))$ must be term equal to 0:

$1 - \ln(x) = 0$

$\ln(x) = 1$

$x=e$

So, $e^{1/e}$ is either a maximum or minimum of the function $x^{1/x}$? Let's put in values of x in the equation for y' to see the derivative curve looks to the left and right of $x=e$.

Again, $y' = (1/x)^2 * (1-\ln(x)) * x^{1/x}$

What if $x=1$? Then $y'=(1/1)^2*(1-\ln(1))*1^{1/1} = 1*1*1 = 1$.

What if $x=e^2$? The first and last terms will be hard to evaluate exactly, but all we need to know is if the whole expression is positive or negative. $(1/x)^2$ and $x^{1/x}$ will both obviously be positive at $x=e^2$. The middle term, $1-\ln(x) = 1-\ln(e^2) = 1-2*\ln(e) = 1-2 = -1$. Thus the whole expression of y' will be negative at a value of e^2.

So, if the derivative curve is positive before $x=e$ and negative after, then $x=e$ must be a relative maximum.

Thus, the function $y = x^{1/x}$ reaches a maximum value at $x=e$.

So $e^{1/e} >= \pi^{1/\pi}$

Going back to the beginning, if $e^{1/e} > \pi^{1/\pi}$, then $e^{\pi} > \pi^{e}$