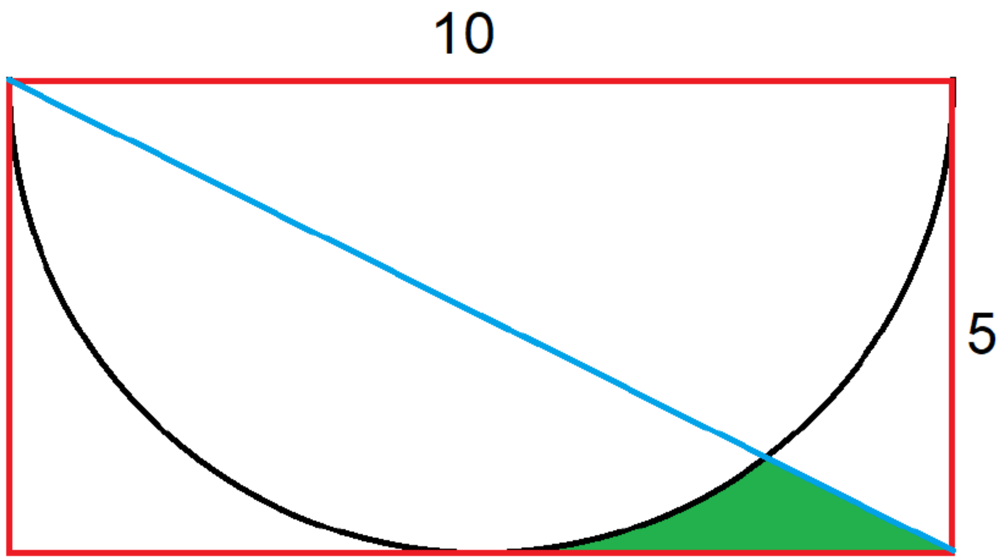
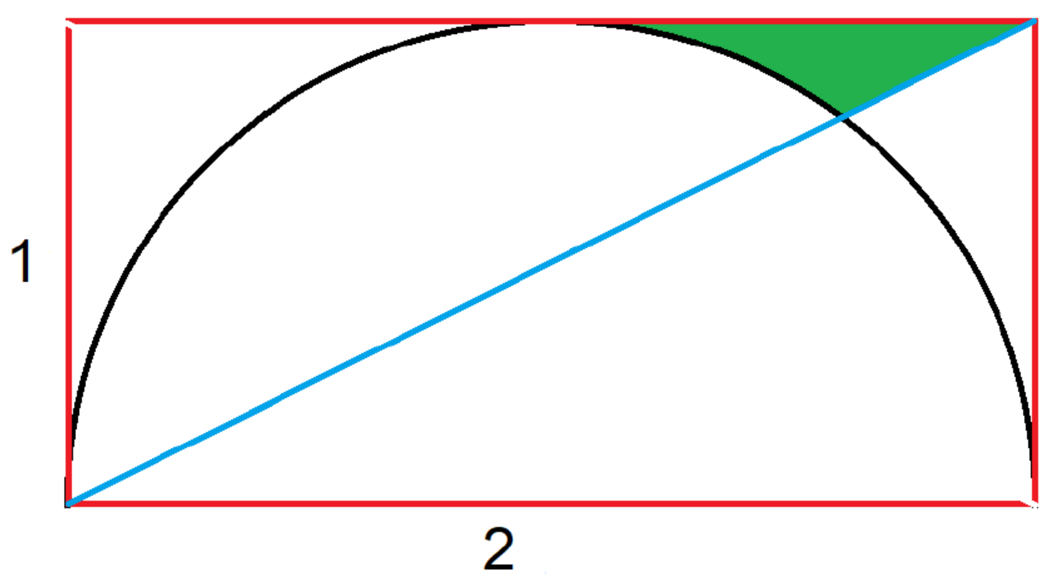
Problem 247

Question: In the following diagram, what is the area of the green region?



Answer:

First, let's change the dimensions to 2 x 1, to simplify the math, reducing the scale of the area by a factor of 10×5 / 2×1 = 25. We will multiply by 25 at the end to scale back up. Second, let's flip over the image, for reasons we'll see soon. That gives us:



Let's use geometry to solve the problem. Let's place the center of the bottom of the rectangle at coordinate (0,0).

The equation for the circle is $x^2 + y^2 = 1$. This illustrates why I wanted to scale down and flip, to get such a simple equation for the circle.

The equation of the blue line is $y = x/2 + 1/2$. Let's rewrite that as $x = 2y - 1$.

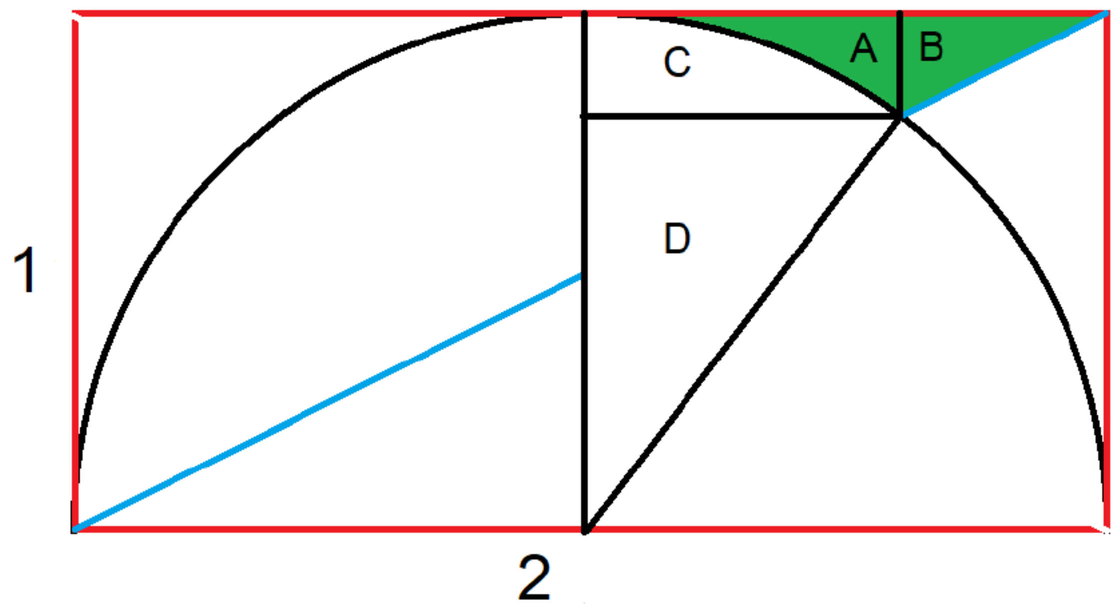
Let's solve for x and y to find where the blue line intersects the semicircle.

$x^2 + y^2 = 1$
$(2y - 1)^2 + y^2 = 1$
$5y^2 - 4y = 0$
$5y - 4 = 0$
$y = 0.8$

Putting that in $x=2y-1$ gives us

$x = 2\times0.8 -1 = 0.6$.

Next, let's label some of the regions in play.



Let's start by find the slice of the semicircle identified as C + D. We already know the coordinate where the blue line crossed the semicircle by the green region is (0.6, 0.8). So, the side of triangle D are 0.6, 0.8, and 1. The area of D is easily found as (1/2)×(0.6 × 0.8) = 0.24.

To find C, let's find the area of the slice of the semicircle C+D and subtract D from it.

The angle of D at (0,0) can be expressed as $\tan^{-1}(3/4)$, $\cos^{-1}(4/5)$, or $\sin^{-1}(3/5)$. I'll arbitrarily decide to go with $\cos^{-1}(4/5)$ =~ 0.6435 (in radians).

The area of the whole circle is pi, divided up by 2×π radians, so the area of the circle (C+D) formed by an angle of $\cos^{-1}(4/5)$ radians is $\cos^{-1}(4/5)/2$ =~ 0.3218.

We subtract D from that slice to get the area of C = $\cos^{-1}(4/5)/2 - 0.24$ =~ 0.0818.

The area of rectangle A+C is 0.2 × 0.6 = 0.12. We know C, so we can find A as 0.12 - [$\cos^{-1}(4/5)/2 - 0.24$] = 0.36 - $\cos^{-1}(4/5)$ =~ 0.0382.

The two legs of triangle B are 0.2 and 0.4, thus the area of B is (0.2 × 0.4)/2 = 0.04.

Thus, the green region is A + B =
$0.36 - \cos^{-1}(4/5) + 0.04 =$
$0.4 - \cos^{-1}(4/5)/2 =\sim 0.0782494$

Remember we scaled the problem down by a factor of 25 at the beginning, so let's scale that area up by a factor of 25, to account for the 5x10 region to begin with, to get an answer of $10 - 12.5\times \cos^{-1}(4/5) =\sim 1.95624$

My thanks to Presh Talwalker for this problem, who in turn gives credit to Xavier in Shanghai. Presh's YouTube channel is Mind Your Decisions. He goes over a solution to this problem at https://www.youtube.com/watch?v=2Seb863FnfU

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