Taylor's Formula says:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Where $f^{(n)}(a)$ = the nth derivative of *f(x)* at x-a.

Let's let a=0, to simplify the formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

We want to find the Taylor series for three formulas: e^x, sin(x), and cos(x).

<u>e^x</u>

Recall that the derivative of of $e^x = e^x$

Taylor's Formula says:

$$e^x = \sum_{n=0}^{\infty} \frac{e^0}{n!}x^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

sin(x)

Using Taylor's formula:

$$\sin(x) = \frac{sin(0)}{0!}x^0 + \frac{cos(0)}{1!}x^1 + \frac{-sin(0)}{2!}x^2 + \frac{-cos(0)}{3!}x^3 + \frac{sin(0)}{4!}x^4 + \frac{cos(0)}{5!}x^5 + \cdots$$

$$= \frac{0}{0!} \quad + \frac{1}{1!}x^1 - \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x \quad - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

cos(x)

Using Taylor's formula:

$$\cos(x) = \frac{cos(0)}{0!}x^0 + \frac{-sin(0)}{1!}x^1 + \frac{-cos(0)}{2!}x^2 + \frac{sin(0)}{3!}x^3 + \frac{cos(0)}{4!}x^4 + \frac{-sin(0)}{5!}x^5 + \cdots$$

$$= \frac{1}{0!} - \frac{0}{1!}x^1 - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 - \frac{0}{5!}x^5 + \cdots$$

$$= 1 \quad - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

The question at hand is to solve for $e^{\pi i}$. Let's start with putting πi in the Taylor series of e^x.

$$e^{\pi i} = 1 + \frac{\pi i}{1!} + \frac{\pi i^2}{2!} + \frac{\pi i^3}{3!} + \frac{\pi i^4}{4!} + \frac{\pi i^5}{5!} + \frac{\pi i^6}{6!} + \cdots$$

$$e^{\pi i} = 1 + \frac{\pi}{1!} i - \frac{\pi^2}{2!} - \frac{\pi^3}{3!} i + \frac{\pi^4}{4!} + \frac{\pi^5}{5!} i - \frac{\pi^6}{6!} + \cdots$$

$$(1) \quad e^{\pi i} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + i \left[\frac{\pi}{1!} - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} \right] \ldots$$

These terms look reminiscent of those for sin(x) and cos(x). Let's look at the Taylor series for sin(π) and cos(π).

$$\sin(\pi) = \pi \quad - \frac{1}{3!}\pi^3 + \frac{1}{5!}\pi^5 - \frac{1}{7!}\pi^7 + \cdots$$

$$\cos(\pi) = 1 \quad - \frac{1}{2!}\pi^2 + \frac{1}{4!}\pi^4 - \frac{1}{6!}\pi^6 + \frac{1}{8!}\pi^8 + \cdots$$

We can conveniently plug these series into equation (1):

$$e^{\pi i} = \cos(\pi) + i \times \sin(\pi) \ldots$$

sin(π) = 0 and cos(π) = -1, so:

$$e^{\pi i} = -1$$

Solution by Michael Shackleford