Question: You are in charge of making cone-shaped paper cups. Your goal is to maximize the ratio of volume to surface area. What is the maximum that ratio can be? Assume the length from the tip of the cup to any point on the edge is 1.

First, let's define some terms. Let:
r = radius of the base of the cone.
h = height of cone
c = circumference of the base of the cone
S = surface area of cone
V = volume of cone

Let's say you make a straight cut from any point on the edge to the tip of the cone. Then you flatten it out. It should make a shape like Pac Man.
Keep thinking about the surface of the cone laid out flat, like a Pac Man shape. If it were not for the slice cut out, the circle would have a radius of 1 and a circumference of 2π. Let's call that the "big circle."

If the radius of the base of the cone is r, then the circumference of the base of the cone is $2r\pi$.

Let's call the angle of Pac Man's mouth θ. The ratio of θ to 2π will be equal to the ratio of the big circles less the base of the cone to the circumference of the base the big circle. In other words

(1) $\theta / 2\pi = (2\pi - 2r\pi) / 2\pi$

The area of slice cut out of Pac Man will equal the ratio of $\theta / 2\pi$ times the area of the big circle:

$(\theta / 2\pi) \times \pi$

Substituting $\theta / 2\pi$ from equation (1) gives us:

$(2\pi - 2r\pi) / 2\pi \times \pi =$
$(2\pi - 2r\pi) / 2 =$
$\pi - r\pi$

So, the area of the slice is $\pi - r\pi$, thus the area of Pac Man is $\pi - (\pi - r\pi) = r\pi$

As a reminder the Pac Man shape is the cone flattened out, so $S = r\pi$.

Next, let's work on an equation for the volume of the cone. Let's integrate up and down the height of the cone.

Let's let y equal any point up the cone. Consider the cone with the point on the bottom. When $y = 0$ the radius of the circle of the cross section will be 0. when $y = h$, the radius of the cross section will be r. So, we need to apply a scaling factor of r/h to the radius of the cross section, so that the radius ranges from 0 to h.

Next, let's solve for h. We know the cone has a lateral length (distance form tip to edge of circumference) of 1 and a radius of r. Using the Pythagorean formula:

$h = \sqrt{1 - r^2}$

As we know, the area of a circle = $\pi \times \text{radius}^2$.

So, the integral for the volume is:

$$V = \int_0^{\sqrt{1-r^2}} \pi \times \left(\frac{\mathrm{ry}}{\sqrt{1-r^2}}\right)^2 \; dy =$$
$$\pi \times \left(\frac{\mathrm{r}}{\sqrt{1-r^2}}\right)^2 \times \int_0^{\sqrt{1-r^2}} \mathrm{y}^2 \; dy =$$
$$\pi \times \left(\frac{\mathrm{r}}{\sqrt{1-r^2}}\right)^2 \times \frac{y^3}{3} \text{ from 0 to } \sqrt{1 - r^2} =$$
$$\pi \times \left(\frac{\mathrm{r}}{\sqrt{1-r^2}}\right)^2 \times \frac{1}{3} \times (1 - r^2)^{3/2} =$$

$\frac{\pi}{3} * \mathrm{r}^2 \; * (1 - r^2)^{-1} * \; (1 - r^2)^{3/2}$ =
$\frac{\pi}{3} * \mathrm{r}^2 \; \times (1 - r^2)^{1/2}$ =

So, we can see the light at the end of the tunnel now. Our equation of the volume to surface area is:

V/S = $\frac{\pi}{3} \times \mathrm{r}^2 \; \times (1 - r^2)^{1/2}$ / rπ =

$\frac{1}{3} \times \mathrm{r} \times \; (1 - r^2)^{1/2}$

Let's maximize the ratio by taking the derivative with respect to r and setting it equal to 0:

f'(r) = $\frac{1}{3} \times (1 - r^2)^{1/2} + \frac{\mathrm{r}}{3} \times (-2r) \times \frac{1}{2} \times (1 - r^2)^{-1/2} = 0$
$\frac{1}{3} \times (1 - r^2)^{1/2} = \frac{\mathrm{r}}{3} \times 2r \times \frac{1}{2} \times (1 - r^2)^{-1/2}$
Multiply both sides by $(1 - r^2)^{1/2}$:

$\frac{1}{3} \times (1 - r^2) \quad = \frac{\mathrm{r}}{3} \times 2r \times \frac{1}{2}$
$(1 - r^2) \quad = r^2$
r^2 = 1/2
r = $\frac{\sqrt{2}}{2}$ =~ 0.7071
The original question was what is the maximum ratio of the volume to surface area.
When r = $\frac{\sqrt{2}}{2}$
S = $\pi \times \frac{\sqrt{2}}{2}$
V = $\frac{\pi}{3} \times \frac{1}{2} \times (1 - \frac{1}{2})^{1/2}$ =

$\frac{\pi}{3} \times \frac{1}{2} \times \frac{\sqrt{2}}{2} =$

$$\frac{\pi \times \sqrt{2}}{12}$$

So V/S =

$\frac{\pi \times \sqrt{2}}{12} / \pi \times \frac{\sqrt{2}}{2} =$

$$\frac{1}{6}$$

This maximum ratio is achieved when

$r = \frac{\sqrt{2}}{2}$

$h = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{2}$

In other words, cut out (or overlap) 25% of the tortilla and form a cone.

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